

MINIMAL AREA NONORIENTABLE STRING DIAGRAMS

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Abstract

We use minimal area metrics to generate all nonorientable string diagrams. The surfaces in unoriented string theory have nontrivial open curves and nontrivial closed curves whose neighborhoods are either annuli or Möbius strips. We define a minimal area problem by imposing length conditions on open curves and on annular closed curves only. We verify that the minimal area conditions are respected by the sewing operations. The natural objects that satisfy recursion relations involving the propagator, which performs both orientable and nonorientable sewing, are classes of moduli spaces grouped by Euler characteristic.

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1 Introduction

In calculating perturbative string amplitudes one must integrate suitable forms over moduli spaces of Riemann surfaces. For this purpose one requires a concrete presentation of each surface in the moduli space, a string diagram, and a rule to build all the string diagrams in a given moduli space. A natural way to achieve this was demonstrated by Zwiebach [1] using minimal area metrics. The uniqueness of the minimal area metric is used to ascertain that each surface is produced once and only once. One can show that whenever surfaces with appropriate minimal area metrics are sewn together, the resulting surface also has a minimal area metric — the so-called sewing theorem. Using this technique all diagrams for a string amplitude can be generated in a natural way.

The purpose of this paper is to extend this work to the previously-unexamined case of surfaces which are not oriented. This case is only natural to consider, since Type I string theory is only consistent with gauge group $SO(32)$, whose fundamental representation is real and which therefore gives rise to nonorientable diagrams [2]. We shall review nonorientable surfaces, then proceed to extend the minimal area problem to include them. We shall see that, while nontrivial open curves are no different from the orientable case, nontrivial closed curves have neighborhoods that are either annuli or Möbius strips, and the minimal area problem imposes length conditions on the annular closed curves and open curves only. It is satisfying that length conditions are thus placed only on curves that correspond to external string states. It is additionally interesting that Möbius curves are forced to have lengths no less than the shortest *open* curves.

We also work through two explicit examples of covering moduli spaces, and finally prove the sewing theorem and establish the nonorientable sewing recursion relations. For a detailed examination of the oriented open-closed case, see [3]. Constraints on conformal field theories existing on nonorientable surfaces can be found in [4, 5].

2 Review of Nonorientable Surfaces

As is well-known, generic orientable surfaces in two dimensions can be characterized topologically by the number of handles (genus) and the number of boundaries. Nonorientable surfaces are characterized additionally by the number of crosscaps. A crosscap is constructed by removing a circle from the surface and identifying opposite points on the boundary.

The familiar Möbius strip can be constructed as a rectangle with one pair of opposite edges identified in a nonorientable fashion. This is similar to the annulus, where the identification is orientable. We can make the single boundary manifest by splitting the Möbius strip down the middle and putting the two halves of the boundary next to each other (Fig. 1). We see that the Möbius strip has one boundary and one crosscap. All surfaces can be rearranged in this way so as to be shown to be constructed out of boundaries, crosscaps and handles. (For example, see [6].)

As the Möbius strip is to the annulus, the Klein bottle is to the torus; the Klein bottle can be thought of as a cylinder with the opposite ends identified nonorientably, or a rectangle

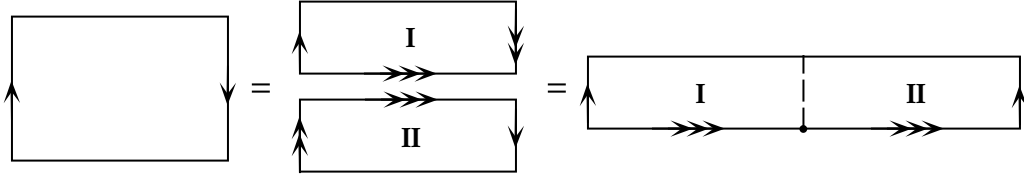


Figure 1: Moving between realizations of the Möbius strip.

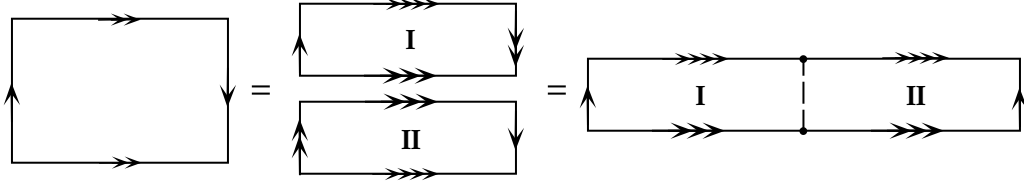


Figure 2: Moving between realizations of the Klein bottle.

with two opposite sides identified orientably and the other pair, nonorientably. Again, we can display the surface in another form to make its crosscap structure manifest (Fig. 2); we see that the Klein bottle is the surface with two crosscaps.

While the torus has a complex conformal Killing vector — corresponding to the translation invariance of the torus inherited from the complex plane — the Klein bottle only has one real conformal Killing vector. In the two-crosscaps realization, this is easy to see; moving around the circumference of the tube is a symmetry, but moving closer to or away from a crosscap obviously is not. In the cylinder realization, the symmetry is translations along the tube, while moving around the circumference is not a symmetry; this is because the nonorientable sewing induces two fixed points on the ends of the cylinder.

The Klein bottle is also different from the torus in the structure of its moduli space. The torus has one complex modular parameter τ , which parameterizes the length of the cylinder and the angle through which it is twisted before being sewn together. The Klein bottle has just a real modulus, corresponding to the length of the cylinder; a twist only returns another equivalent Klein bottle. Furthermore, the Klein bottle has no modular group. The transformation $\tau \rightarrow -\frac{1}{\tau}$ produces a distinct Klein bottle, because unlike the toral case where the two pairs of sides are both identified in the same way, the two pairs of sides of the Klein bottle are identified in different ways. The moduli space of the Klein bottle is thus $(0, \infty)$.

For general surfaces, crosscaps, handles and boundaries all modify the Euler characteristic χ :

$$\chi = 2 - 2g - c - b - n - \frac{m}{2} \quad (1)$$

where g is the genus, c the number of crosscaps, b the number of boundaries, n the number of punctures on the surface and m the number of punctures on the boundary components. This in turn dictates the number of moduli of the surface

$$\dim \mathcal{M}_{b,m}^{g,c,n} = 6g - 6 + 2n + 3b + 3c + m. \quad (2)$$

In all cases the number of crosscaps enters the formulae with half the coefficient of the genus. This is due to a theorem stating that in the presence of a crosscap, a handle can always be decomposed into two more crosscaps: [6]

$$\text{handle} + \text{crosscap} = 3 \text{ crosscaps}. \quad (3)$$

Hence, when $c \neq 0$ the number of handles can be made zero and crosscaps, boundaries and punctures then specify the surface topologically.

In contrast to the case of oriented closed string theory, at each order in perturbation theory several different moduli spaces contribute to a generic amplitude in nonorientable open-closed string theory. This occurs because boundaries and crosscaps can also appear and contribute powers of the coupling, resulting in different topologies at the same order of the expansion. Looking at string-scattering arguments, one can see that a crosscap contributes κ , the gravitational coupling constant, to the order in perturbation theory of a diagram. One can then establish how many different moduli spaces enter the computation of a given amplitude at each order.

Since the contribution to the coupling from external states is the same for a given amplitude at all orders, we can ignore this factor for the purposes of seeing how many diagrams arise at each level. Then we can concentrate on the factor of the coupling κ^l by which a given diagram differs from the sphere with the same number of punctures. A diagram with open-string punctures has the additional requirement that there be at least one boundary.

Let us first look at oriented open-closed string theory. Any diagram contributing to the process at $\mathcal{O}(\kappa^l)$ must have a genus g and b boundary components satisfying

$$l = 2g + b. \quad (4)$$

For external closed string states only, $\frac{l+2}{2}$ moduli spaces contribute for l even and $\frac{l+1}{2}$ for l odd. If there is at least one external open string, l even has only $\frac{l}{2}$ distinct moduli spaces, since we must have $b \neq 0$. The result for l odd is unchanged.

For the nonorientable sector, we can use the relation between crosscaps and handles to characterize the surface solely in terms of crosscaps c , boundaries b and punctures. Then diagrams of $\mathcal{O}(\kappa^l)$ have

$$l = c + b \quad (5)$$

and there are l distinct moduli spaces at this order, where we have not included $c = 0$ as this falls in the orientable sector. Hence there are l possibilities if we can have $b = 0$ and $l - 1$ if $b \neq 0$. For the nonorientable theory in general, both sectors contribute.

3 The Minimal Area Metric

The length conditions of the minimal area problem for orientable closed and open strings are that nontrivial closed curves must have at least length 2π and nontrivial open curves must have length at least π [1]. The first number is arbitrary and is chosen to correspond to the circumference of the semiinfinite cylinders that represent external closed strings. The ratio of the values of the two conditions is a variable parameter that defines a one-parameter family of string field theories; the choice above corresponds to a theory where every open/closed string diagram has as its double a closed string diagram [7]. For nonorientable surfaces, these length conditions are no longer adequate, due to the appearance of a new class of closed curve, the “Möbius” curve. We will examine how this curve arises, and we will be led naturally to the new length condition that must be imposed.

Consider a closed curve γ and consider a neighborhood of γ along its length — define two curves on either side of γ that are locally parallel to it by taking the set of all points a fixed distance along the normal vectors to γ , and consider all the points between these two flanking curves. We can think of the curve as having been “fattened” into a two-dimensional submanifold with boundary. If this fattened curve is topologically a Möbius strip, then γ is a Möbius curve. If, on the other hand, the fattened curve is a simple annulus, the curve is the ordinary orientable type we are used to, which we shall call an annular curve.

To see how the curves differ, let us examine the Möbius strip in both of the realizations discussed previously. First, consider the classic realization of the Möbius strip as a strip that is twisted and attached to itself. A curve traveling along the strip parallel to the boundary will generically loop around twice before returning to itself. However, a curve precisely at the center of the strip will come back to itself after only one loop (Fig. 3). This curve, by definition, is a Möbius curve. To see that the other curve is annular, look at these same curves in the crosscap realization. Now the longer curve is seen to encircle the crosscap without touching it — it fattens to an annulus. The Möbius curve is seen to pass through the crosscap. We see that the shortest Möbius curves are just half the length of the shortest annular curves — which interestingly is the same length as the shortest nontrivial open curves.

What would happen if we kept the same minimal area problem that was used for orientable surfaces? The closed-curve length conditions would apply to all nontrivial closed curves, Möbius and annular alike. Consider a cylinder ending on a crosscap. In order to keep Möbius curves going through the crosscap at least length 2π , the crosscap itself must have circumference 4π . However, the cylinder leading to the crosscap need not have circumference any greater than 2π , and to minimize area it will reduce to this size (Fig. 4). Viewed in the Möbius strip realization, the surface is more pathological. There is no sensible way to interpret a diagram such as this in terms of strings.

Thus, the orientable length conditions are in need of modification for the nonorientable case. The natural thing to do is to require that nontrivial annular closed curves have length 2π , while Möbius curves need only have length π , and the same for the nontrivial open curves. In fact, as we shall prove below, it is sufficient only to set conditions on the annular

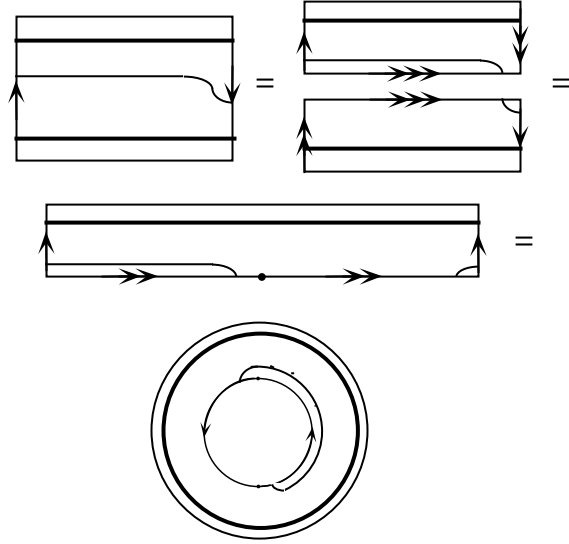


Figure 3: Two different classes of closed curve as seen on the Möbius strip. The heavy curve is a familiar annular curve, while the lighter curve (deformed away from its minimal length for clarity) is the Möbius curve.

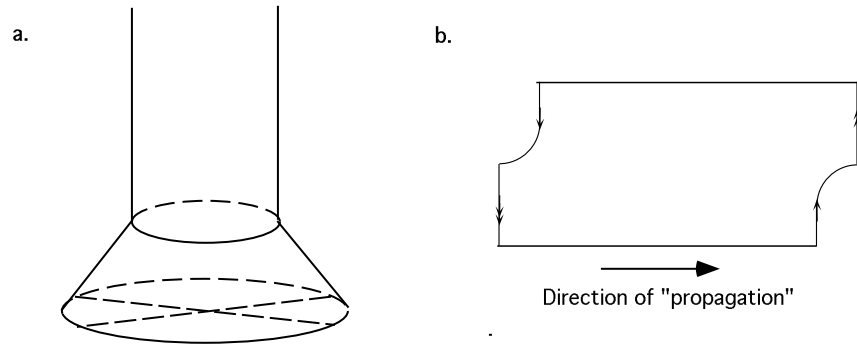


Figure 4: A string ending in a crosscap when the naive length conditions established for the orientable case are applied to all curves on a nonorientable surface. In a. at left, the string must expand to end in the crosscap; in b. at right, in the alternate realization, there is no interpretation in terms of propagating strings.

closed and the open curves; the length conditions on the Möbius curves follow.

To see this, consider a crosscap and a Möbius curve α_1 , wrapped around half the crosscap, and thus as short as it can be. Impose the conditions that nontrivial annular curves have length of at least 2π , but impose no condition on the Möbius curves. Let α_1 have a length less than π ; we seek to show that then there must exist a nontrivial annular curve with length less than 2π , in contradiction with our assumptions. There is another Möbius curve, α_2 , which goes around the other half of the crosscap. It is clear that the smallest annular curve that can surround the crosscap γ has a length equal to the sum of the two Möbius curves. If the metric on the α_2 side of the crosscap is the same as the metric at α_1 — in other words, if the metric is continuous across the crosscap — the two have the same length and their sum, and thus the annular curve's length, is less than 2π , our desired contradiction. Imagine that the metric is not continuous across the crosscap, and thus that α_1 has different (say, shorter) length than α_2 . The metrics on either side of the crosscap in general vary with position, but since $l_{\alpha_1} < l_{\alpha_2}$, there must be at least one point on the 2 side with metric greater than at least one point on the 1 side. We can lower the metric on the 2 side at this point while raising it on the 1 side in such a way as to preserve $l_\gamma = l_{\alpha_1} + l_{\alpha_2}$; the perturbation is simply

$$\rho_1 \rightarrow \rho_1 + \delta\rho \tag{6}$$

$$\rho_2 \rightarrow \rho_2 - \delta\rho \tag{7}$$

where $ds^2 = \rho^2 dzd\bar{z}$, $\delta\rho > 0$ and the metric deformations take place in squares of infinitesimal side length ϵ centered on the points in question. The total area is quadratic in the metric, however, and changes as

$$\mathcal{A} \rightarrow \mathcal{A}' = \mathcal{A} + 2\delta\rho\epsilon^2(\rho_1 - \rho_2) + \mathcal{O}(\delta\rho^2) < \mathcal{A}. \tag{8}$$

We have found a metric perturbation that preserves length conditions, but lowers the total area. Thus the metric configuration with a discontinuity across a crosscap is not of minimal area.

We see placing conditions on the annular curves is sufficient to induce the appropriate conditions on the Möbius curves as well. This is satisfying, since we need only place conditions on curves that correspond to propagators and external string states.

We are now prepared to state the minimal area problem for nonorientable open/closed strings:

Minimal Area Problem for Nonorientable Open-Closed String Theory: Given a genus g surface R with b boundaries, c crosscaps, m punctures on the boundaries and n punctures in the interior ($g, b, c, n, m \geq 0$) the string diagram is defined by the metric of minimal (reduced) area under the condition that the length of any nontrivial open curve in R with endpoints at the boundaries be greater than or equal to π and the length of any nontrivial *annular* closed curve be greater than or equal to 2π .

4 Two explicit examples

Diagrams are constructed from vertices and propagators, sewing together surfaces with fewer moduli to span part of moduli space, and covering the rest by adding the necessary diagrams to that moduli space's vertex. To illuminate the methods, we will construct the diagrams for two different cases. First we will examine the Klein bottle, which is interesting as a nonorientable version of the torus. Then, for a more complicated example, we will look at the Klein bottle with boundary, a surface with two crosscaps and one boundary.

The Klein Bottle

As we have discussed, the Klein bottle is a zero-genus surface with two crosscaps and no boundaries. We begin by constructing all possible Klein bottles from more elementary diagrams and seeing how much of moduli space we span this way. We can self-sew arbitrarily long closed-string propagators nonorientably, and thus obtain Klein bottles at various points in moduli space all the way to infinity (Fig. 5a). However, we cannot make the propagator too short, for if its length is less than π , the Möbius closed curve around its middle will violate the length conditions. It becomes necessary for us to go to another “channel” to span the part of moduli space with small values of the parameter.

The other way to make a surface with two crosscaps is to sew together two closed string self-interaction vertices, with an arbitrarily long closed string propagator in the middle (Fig. 5b). Viewing this two-crosscaps realization in the more familiar self-sewn cylinder realization, we see that very long two-crosscap diagrams correspond to cylinders of length π (satisfying length conditions) but with arbitrarily large circumferences. Since it is the ratio of length to circumference that matters, these correspond to arbitrarily small values of the modular parameter.

Thus we have filled in both ends of moduli space. Additionally, we see that the two channels come together smoothly in the middle, at the surface which is a cylinder of length π and circumference 2π , saturating the length conditions (Fig. 5c). We have now spanned moduli space for the Klein bottle successfully.

Klein bottle with boundary

This is a considerably more complicated surface, and we will not describe the process for spanning moduli space in quite the same detail. There are three real moduli. These can be thought of as

1. the length of the Klein bottle
2. the size of the boundary, and
3. the location of the boundary.

The location of the boundary is one real parameter because the Klein bottle has one real conformal Killing vector, rendering the location of the boundary in one of the two dimensions

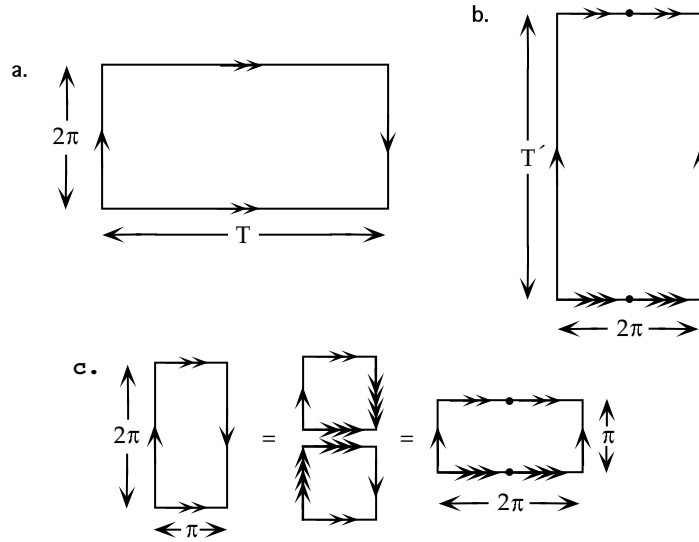


Figure 5: Spanning the moduli space of the Klein bottle. Large values of the modular parameter are obtained by self-sewing an arbitrarily long propagator nonorientably, as in a. Small values are achieved in the other realization of the Klein bottle, by connecting two crosscaps with a propagator of arbitrary length, as in b. The two channels come together smoothly when the surfaces saturate their length conditions, at $T = T' = \pi$, as shown in c.

of the surface irrelevant. The location in the other direction is relevant; in fact, movement along this direction is equivalent to a twist of the bottle.

There are six ways of constructing the surface with two crosscaps and a boundary; they are displayed in Figure 6. It is not hard to show that together they fill in the “corners” of moduli space, that is, that the asymptotic regions of moduli space can be covered by allowing the various propagators to get very large. Once again what is large in one channel corresponds to a small value of moduli in another.

It is a little bit of work to show that the six all come together smoothly in the middle. The large-boundary and small-boundary diagrams (1 and 3, and 2 and 4, respectively) interpolate into one another smoothly as shown in Figure 7. To connect the first and second diagrams, for example, we look at them both in the same realization and see what happens to the boundary when the Klein bottle saturates its length conditions (Fig. 8). It is most naturally reduced to a slit in each case, but the slits are at right angles to each other and in different locations. The location of one slit can be shifted to the other by twisting the Klein bottle. In order for the boundaries to interpolate into each other, we must have instead of the naive slit a progression through a square as shown in Figure 9. This connects the first four diagrams; the last two can be done similarly.

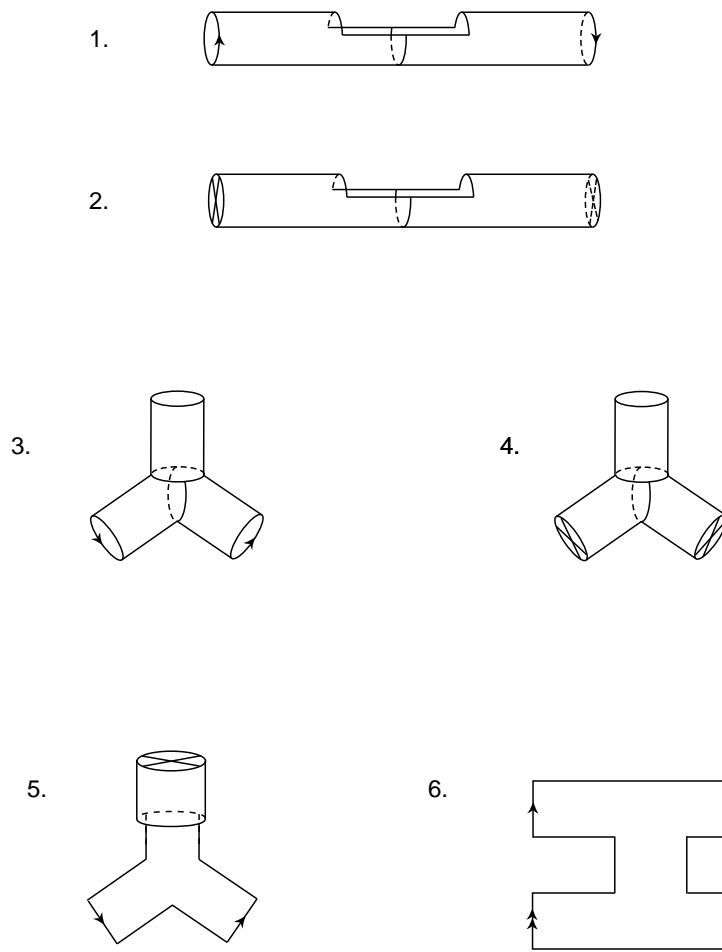


Figure 6: The six ways of constructing the Klein bottle with boundary .

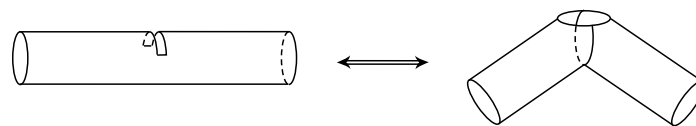


Figure 7: The interpolation between large- and small-boundary diagrams.

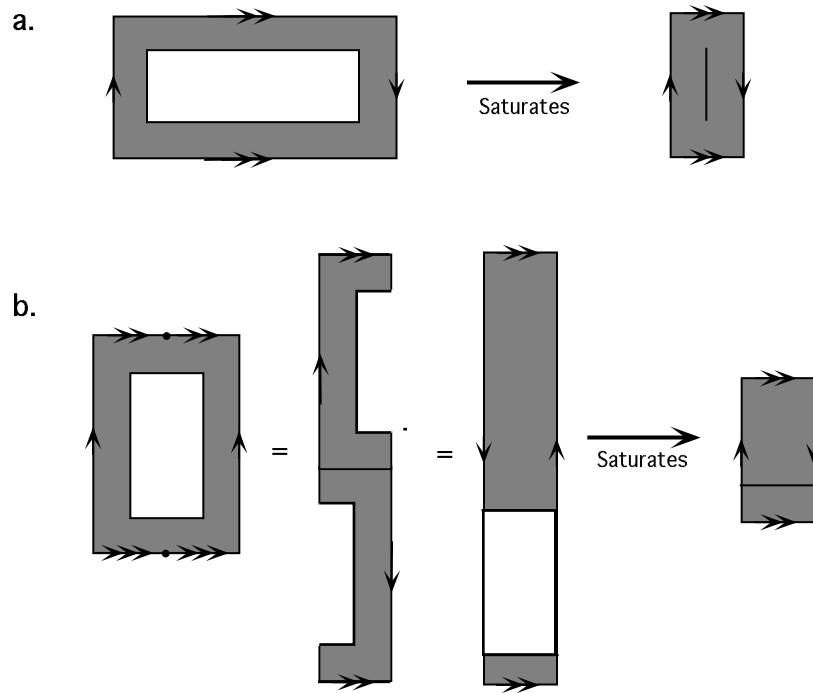


Figure 8: The surfaces identified as 1 and 2 on Figure 6 saturating the length conditions on the Klein bottle. The surfaces are shaded to keep the location of the boundary clear.

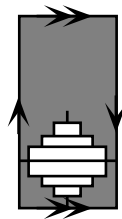


Figure 9: The actual evolution of the boundary as surfaces 1 and 2 come together.

5 Sewing Theorem and Sewing Recursion Relations

The Sewing Theorem

Here we generalize the sewing theorem of Wolf and Zwiebach [8]. One wants to assure that when two surfaces which have minimal area metrics satisfying the length conditions (“admissible”) are sewn together, or when a single surface is sewn to itself, the resulting surface also has an admissible minimal area metric. Thus we must ascertain that after the process of sewing there are no nontrivial curves which violate the length conditions. The result turns out to be a straightforward extension of the sewing theorem for orientable surfaces. There are three cases to consider.

First consider the curves which do not pass through the sewing line (the line $|z| = |w| = 1$ when sewing with $zw = 1$), and which were nontrivial prior to sewing. The process of sewing only involves two steps: amputating the semiinfinite cylinders/strips that represent legs which are to be sewn, and then identifying the stubs. But the metric used after amputation ρ_0 is just the restriction of the original metric ρ to the amputated surface; hence all the curves on the surface retain the same length. Since sewing is then simply a process of identification, which does not affect the lengths of curves, we see that all curves that satisfy the length conditions before sewing, continue to do so after sewing, regardless of type.

Now we must consider the curves which were trivial before sewing. These also retain the same length, but if a short one were to become nontrivial in the sewing process, the length conditions would be violated. A trivial closed curve γ must bound a disc D , regardless of whether the surface is orientable or nonorientable. (The proof uses the universal covering space, and is not modified much for the nonorientable case, since every nonorientable two-dimensional surface has an orientable surface as its cover [6]. For the proof on Riemann surfaces, see [9].) However the process of sewing does not affect this disc; sewing only connects two punctures, and since D cannot contain a puncture and still be a disc, it remains a disc after the sewing. Thus γ remains trivial. A trivial open curve also can be seen to bound half a disc. Any trivial open curve must begin and end on the same boundary component. If the points on the boundary where the curve β begins and ends are A and B , then β and line segment \overline{AB} must bound a disc. The same argument as above then shows that the curve must remain trivial. There are no trivial Möbius curves; a trivial curve must bound a disc, but if we “fatten” up a curve bounding a disc, we obtain an annulus, not a Möbius strip.

Now we examine the case of curves that pass through the seam. A curve that is homotopic to the seam must at least be the length of the seam itself, and the seam satisfies the length conditions, so these curves are safe. For curves not homotopic to the seam, we have used the choice of where to amputate the semiinfinite cylinders and strips to our advantage. We deliberately keep stubs of length π to ensure that the length conditions are satisfied in the new surface. The curves must extend out of the cylinder/strip and into the rest of the surface, where there are other punctures, boundary components, cycles etc. which are necessary to make them nontrivial. Thus the curve must traverse either one of the stubs twice or each stub once, acquiring thereby a length of no less than 2π . Thus it satisfies the length conditions regardless of the type of curve.

Hence the sewing of two surfaces with admissible metrics always produces another surface with an admissible metric, and the sewing theorem is complete.

Sewing Recursion Relations

Different classes of diagrams cover different regions of moduli space, and for the complete covering to be sensible, recursion relations must hold at the boundaries of the different regions. Specifically, the vertex covers a certain region of moduli space; the string diagrams on the boundary of this region must match with diagrams created from more elementary vertices with one collapsed propagator. This is expressed by the master equation

$$\partial\mathcal{V} + \frac{1}{2}\{\mathcal{V}, \mathcal{V}\} + \hbar\Delta\mathcal{V} = 0 \quad (9)$$

where the antibracket operation $\{ , \}$ sews together two different surfaces and Δ sews a single surface to itself, and the full vertex \mathcal{V} will be defined below. For details, see [3]. Each of these operations contains all possible relevant types of sewing, closed and open. In considering the nonorientable case, one has an additional new way to sew, nonorientable sewing.

When sewing two generic surfaces with coordinates z and w in the neighborhoods of the punctures to be connected, one sews by identifying the coordinates: $zw = \text{const}$. However nonorientability permits a new kind of identification: $z\overline{w} = \text{const}$. This “antiholomorphic” identification produces a nonorientable surface when an orientable surface is self-sewn. Holomorphic and antiholomorphic coordinates on the same surface are linked; orientability is thus destroyed, as a small oriented circle (indicatrix) can pass through the seam along a closed curve and return with its orientation reversed.

Consider sewing two disjoint surfaces in an antiholomorphic fashion. Sewing surface A to surface B in this way is equivalent to holomorphically sewing surface A to the mirror of surface B , that is, the surface resulting from exchanging the holomorphic and antiholomorphic coordinates on B . But the mirror of B is just another surface, with the same values of g , b , c , etc. as B itself, and already present in the collection of surfaces under consideration. Thus to sew two disconnected surfaces antiholomorphically produces the same resulting surface as a different holomorphic sewing. Hence if all surfaces are sewn both holomorphically and antiholomorphically, the antibracket operation will produce each surface exactly twice.

The propagator is fixed, regardless of which surfaces are being sewn. Hence we require a single prescription incorporating both kinds of sewing:

$$\text{Propagator} \equiv \frac{1}{2}(sew + \overline{sew}) \quad (10)$$

where sew and \overline{sew} denote holomorphic and antiholomorphic sewing, respectively. The factor of $\frac{1}{2}$ ensures that this prescription will produce each surface once when different surfaces are sewn via the $\{ , \}$ operation. The Δ operation will then produce two different resulting surfaces for each surface it is applied to, with the factor of $\frac{1}{2}$.

Sewing operations change the topology of the surface(s) on which they are performed. The orientable sewing operations have been enumerated in [3]. Antiholomorphic sewing of

String type sewn	Surfaces Sewn	Boundaries Sewn	Orientation	Change in Topology
Closed	Different	—	sew, \overline{sew}	—
Closed	Same	—	sew	$g \rightarrow g + 1$
Closed	Same	—	\overline{sew}	$c \rightarrow c + 2$
Open	Different	Different	sew, \overline{sew}	$b \rightarrow b - 1$
Open	Same	Same	sew	$b \rightarrow b + 1$
Open	Same	Same	\overline{sew}	$c \rightarrow c + 1$
Open	Same	Different	sew	$g \rightarrow g + 1$ $b \rightarrow b - 1$
Open	Same	Different	\overline{sew}	$c \rightarrow c + 2$ $b \rightarrow b - 1$

Table 1: How each distinct sewing operation affects the topology of the resulting surface. For sewing of different surfaces, g denotes $g_1 + g_2$, etc.

different surfaces has precisely the same topological consequence as its holomorphic counterpart. Antiholomorphic sewing of the same surface produces crosscaps, unique to the nonorientable case. These operations are summarized in Table 1, for a surface with genus g , b boundaries, c crosscaps, and n closed and m open punctures.

The sewing recursion relations on a vertex from a single moduli space do not involve the full form of the propagator, which includes both kinds of sewing on an equal footing. For the Δ operation the two types of sewing change the topology of the surface in different ways, and hence on the boundary of a given vertex it is a different surface which appears with a holomorphically sewn collapsed propagator than the one that appears with an antiholomorphically collapsed propagator.

One can remedy this situation by collecting the various vertices corresponding to surfaces with fixed values of n and m and a given Euler characteristic (or equivalently, the various vertices of a given amplitude at a certain order of the coupling constant) into a single vertex $\mathcal{V}_{n,m}^{\bar{\chi}}$:

$$\mathcal{V}_{n,m}^{\bar{\chi}} = \sum_{g,c,b} \mathcal{V}_{n,m}^{g,c,b} \quad (11)$$

where the summation is constrained to sum over all possible values of g, c, b that produce the appropriate value of $\bar{\chi}$, the Euler characteristic of the surface composing the vertex. The recursion relations applied to the objects $\mathcal{V}_{n,m}^{\bar{\chi}}$ involve the full propagator: although the two different kinds of sewing change the topology in different ways, they modify the Euler characteristic of the surface by the same amount, as one can verify with the help of Table 1.

The total vertex with appropriate values of the coupling κ and \hbar is then:

$$\mathcal{V} \equiv \sum_{n,m,\bar{\chi}} \hbar^{-\bar{\chi}+\bar{p}} \kappa^{-2\bar{\chi}} \mathcal{V}_{n,m}^{\bar{\chi}} \quad (12)$$

where $\bar{p} = 1 - \frac{1}{2}(n + m)$ [3]. Implicit in the sum over open punctures m is a sum over all possible distinct distributions of the punctures over the various boundary components. This vertex then satisfies the master equation, Eq. 9.

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